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Moderately large flexural vibrations of composite plates with thick layers

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Dedicated to Professor Arthur W. Leissa

Abstract

In this paper moderately large amplitude vibrations of a polygonally shaped composite plate with thick layers are analyzed. Three homogeneous and isotropic layers with a common Poisson's ratio are perfectly bonded and their arbitrary thickness and material properties are symmetrically disposed about the middle plane. Mindlin–Reissner kinematic assumptions are implemented layerwise, and as such model both the global and local response. Geometric nonlinear effects arising from longitudinally constrained supports are taken into account by Berger's approximation of nonlinear strain–displacement relations. Overall cross-sectional rotations are defined and subsequently a correspondence of this complex problem to the simpler case of a homogenized shear-deformable nonlinear plate with effective stiffness and hard hinged boundary conditions is found. The nonlinear steady-state response of composite plates subjected to a time-harmonic lateral excitation is investigated and the phenomena of nonlinear resonance are studied and evaluated.

Keywords: Layered plate; Layerwise theory; Geometric nonlinearities; Berger's approximation

1. Introduction

Structural elements such as beams, plates and shells composed of two or more layers are frequently used in various engineering applications. In the last decades, numerous theories have been developed in an effort to predict deformation and stresses of these elements, see e.g. Leissa (1981). In general there exist two classes of theories to describe approximately the kinematics and stress states of layered structures. In the so-called equivalent-single-layer theories the in-plane displacements and their derivatives with respect to the lateral coordinate are continuous through the thickness, and effective material and plate properties are introduced by means of homogenization (e.g. Whitney and Pagano, 1970; Reddy, 1984; Gordaninejad and Bert, 1989; Suzuki et al., 1996). Alternatively, admitting a separate displacement field within the individual layers of the composite leads to layerwise theories (e.g. Di Taranto, 1965; Yan and Dowell, 1972; Adam, 2001). A comparative study of different theories of laminate plates is given by Reddy (1997). Based on von

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Karman's nonlinear plate equations (von Karman, 1910) several modified theories for composite plates have been developed in an effort to take into account moderately large deformations (e.g. Reddy and Chao, 1981; Shi et al., 1997; Ribeiro and Petyt, 1999). An approximation of von Karman's equations has been introduced by Berger (1955) and, subsequently, applied to various homogeneous and composite plate and shallow shell theories (e.g. Heuer et al., 1992; Heuer, 1994; Heuer and Ziegler, 1998). It is commonly accepted that Berger's geometrically nonlinear plate equations gives satisfactory results when the plate edges are prevented from in-plane motion. A comprehensive treatment of nonlinear theories for layered plates including a review of corresponding literature can be found in Sathyamoorthy (1998).

Following a procedure employed for linear composite plates by Heuer et al. (1997) and Adam (2001) the scope of the presented paper is to derive a unifying representation of the influence of moderately large amplitudes on vibrations of a layered plate with polygonal planform. The edges of the considered plate are assumed to be prevented from in-plane motions and are hard hinged supported. Berger's approximation of nonlinear strain–displacement relations are applied to take into account geometrical nonlinear effects arising from longitudinally constrained supports. The homogeneous layers are assumed to consist of isotropic linear elastic materials. According to layerwise theories, Mindlin–Reissner kinematic relations are applied separately to each individual layer. In the special case of a laminated plate composed of isotropic thick layers with physical properties symmetrically disposed about the middle plane, a correspondence to a nonlinear moderately thick homogeneous plate is found. Besides the theoretical interest in such a correspondence the present formulation can be used for generating benchmark solutions in the context of the finite element method.

For a qualitative study, the nonlinear steady-state response of layered plates subjected to time-harmonic lateral excitation is investigated.

2. Governing equations for a three-layer plate

In the present analysis a plate composed of three isotropic and homogeneous layers of moderate thickness is considered. All layers are perfectly bonded and their arbitrary thicknesses and linear elastic properties are symmetrically disposed about the middle plane. The plate is referenced to a Cartesian system of coordinates x, y, z, where the xy-plane ($z = 0$) is the middle plane, and z is a coordinate perpendicular to that plane. The layers may exhibit strongly different elastic moduli with a common Poisson's ratio ν, and therefore, Mindlin–Reissner kinematic assumptions of shear-deformable plates are applied to each layer separately to derive the field equations.

The displacement field of the ith layer is assumed to be of the form, Yu (1995)

$$ {}_{i}u_j = {}_{i}u_j^{(0)} + z\,{}_{i}\psi_j, \quad w_i = w, \quad i = 1, 2, 3, \quad j = x, y, \tag{1} $$

where ${}_{i}u_x$, ${}_{i}u_y$ represent in-plane displacements of the ith layer in x- and y-direction, respectively, at distance z from the xy-plane, ${}_{i}u_x^{(0)}$, ${}_{i}u_y^{(0)}$ denote in-plane displacements at $z = 0$ and ${}_{i}\psi_x$, ${}_{i}\psi_y$ are layerwise cross-sectional rotations, $i = 1, 2, 3$ (see Fig. 1). The index $i = 2$ refers to quantities of the core, while $i = 1, 3$ belong to the upper and lower face, respectively, w denotes the lateral deflection of the plate common to all layer planes. The components ${}_{i}u_j^{(0)}$ of the faces ($i = 1, 3$) can be expressed in terms of ${}_{2}u_j^{(0)}$ and the cross-sectional rotations in order to satisfy the interface displacement continuity relations, Fig. 1, according to

$$ {}_{i}u_j^{(0)} = {}_{2}u_j^{(0)} + z_i({}_{2}\psi_j - {}_{i}\psi_j), \quad i = 1, 2, 3, \quad j = x, y. \tag{2} $$

In (2) $z_1 = -h_2/2$, $z_3 = h_2/2$ denote vertical coordinates from the middle plane to the interfaces between the core and the upper and lower face, respectively.

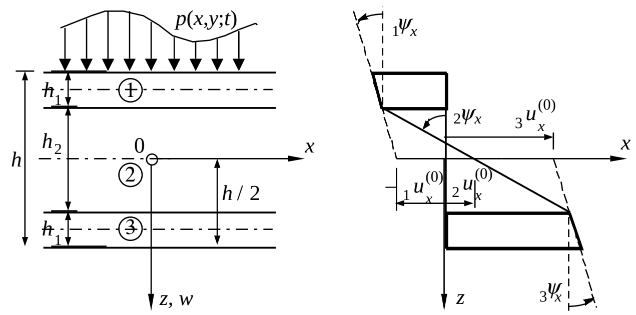

Fig. 1. Three-layer composite plate, *xz*-plane, and corresponding horizontal displacement field.

In this study it is assumed that the transverse deflections of a plate are not small compared to its thickness, and the interaction between the membrane stresses and the curvatures must be considered. This interaction results in the stretching of the middle plane, which leads to nonlinear terms in the strain–displacement relations. For moderately large deflections w of a plate these nonlinear relations, leading to a geometrically nonlinear formulation of the governing plate equations, are given by, see e.g. Sathyamoorthy (1998),

$$e_j = {}_2u^{(0)}_{j,j} + \frac{1}{2}(w_{,j})^2, \quad j = x, y, \tag{3.1}$$

$$e_{xy} = {}_2u^{(0)}_{x,y} + {}_2u^{(0)}_{y,x} + w_{,x}w_{,y}, \tag{3.2}$$

where e_x, e_y are the middle plan strains and e_{xy} is the in-plane shear strain. Hence, the strains at any point away from the middle plane at a distance z become

$$_{\mathrm{i}}\varepsilon_j = {}_{\mathrm{i}}u_{j,j} + \frac{1}{2}(w_{,j})^2, \quad {}_{\mathrm{i}}\gamma_{xy} = {}_{\mathrm{i}}u_{x,y} + {}_{\mathrm{i}}u_{y,x} + w_{,x}w_{,y}, \quad {}_{\mathrm{i}}\gamma_{jz} = {}_{\mathrm{i}}u_{j,z} + w_{,j}, \quad i = 1,2,3, \ \ j = x, y. \tag{4}$$

In (3) and (4) $()_{,j}$ indicates spatial differentiation of $()$ with respect to $j : \partial()/\partial j$.

The constitutive relations, however, are linear. It is assumed that the normal stress component σ_z is negligible and can be dropped. For an isotropic, elastic material the stress components σ_x, σ_y, τ_{xy} are related to the strains by means of the Hooke's law, see e.g. Ziegler (1998),

$$_{\mathrm{i}}\sigma_j = \frac{2G_i}{1-\nu}({}_{\mathrm{i}}\varepsilon_j + \nu\,{}_{\mathrm{i}}\varepsilon_k), \quad {}_{\mathrm{i}}\tau_{xy} = G_i\,{}_{\mathrm{i}}\gamma_{xy}, \quad i = 1,2,3, \ \ j = x, \ \ k = y \text{ and } j = y, \ \ k = x, \tag{5}$$

in which G_i is the shear modulus of the homogeneous ith layer.

Transverse shear stress components τ_{xz}, τ_{yz} are specified to be continuous across the interfaces. Two types of approximations are acknowledged in the literature. If the "correct" shear stress, which is expressed by means of the law of conservation of momentum, is used, the in-plane equilibrium is automatically satisfied, see e.g. Swift and Heller (1974), Yu (1995). Alternatively, prescribing the continuity of the transverse shear stresses according to Hooke's law derives a simplified boundary value problem, Yan and Dowell (1972), Durocher and Solecki (1976) and Heuer (1992)

$$_{\mathrm{i}}\tau_{jz} = G_i({}_{\mathrm{i}}\psi_j + w_{,j}) = {}_{i+1}\tau_{jz} = G_{i+1}({}_{i+1}\psi_j + w_{,j}), \quad i = 1,2, \ \ j = x, y. \tag{6}$$

In analogy to the Mindlin theory for homogeneous plates Eq. (6) exhibits the simplified assumption that the shear stress is uniformly distributed throughout the layer. From this equation it follows that the cross-sectional rotations of both faces are equal

$$ {}_1\psi_j = {}_3\psi_j, \quad j = x, y, \tag{7} $$

since both faces are composed of the same material ($G_1 = G_3$).

Layerwise stress resultants are determined by integration of the stress components, Eq. (5), over the thickness of the layers. At first, the cross-sectional rotations of the faces are eliminated by means of Eq. (6), and hence, the layerwise resultants can be expressed in terms of the lateral deflection w, the cross-sectional rotations ${}_2\psi_x$, ${}_2\psi_y$ of the core, the middle plane strains and their derivatives. The plate stress resultants are subsequently determined by layerwise summation. The overall bending moments ${}_im_x$, ${}_im_y$, ${}_im_{xy}$ and shear forces ${}_iq_x$, ${}_iq_y$ of the composite plate are obtained as

$$ m_j = \frac{2}{1-\nu}\sum_{i=1}^{3}\left\{(\beta_i + C_i)_2\psi_{j,j} + \beta_i w_{,jj} + \nu[(\beta_i + C_i)\,{}_2\psi_{k,k} + \beta_i w_{,kk}]\right\}, \quad j = x,\ k = y \text{ and } k = x,\ j = y, \tag{8.1} $$

$$ m_{xy} = \sum_{i=1}^{3}\left[(\beta_i + C_i)({}_2\psi_{x,y} + {}_2\psi_{y,x}) + 2\beta_i w_{,xy}\right], \tag{8.2} $$

$$ q_j = \delta({}_2\psi_j + w_{,j}), \quad j = x, y, \tag{9} $$

where A_i, B_i, C_i and S_i are the layerwise stiffness coefficients determined by

$$ A_i = G_i h_i, \quad B_i = \tfrac{1}{2}G_i(a_{i+1}^2 - a_i^2), \quad C_i = \tfrac{1}{3}G_i(a_{i+1}^3 - a_i^3), \quad S_i = \kappa^2 G_i h_i, \tag{10.1} $$

$$ a_1 = -h/2, \quad a_2 = -h_2/2, \quad a_3 = h_2/2, \quad a_4 = h/2 \tag{10.2} $$

and

$$ \delta = 2S_1\frac{G_1}{G_2} + S_2, \quad \beta_1 = \beta_3 = \left(\frac{1}{2}B_1 h_2 + C_1\right)\left(\frac{G_2}{G_1} - 1\right), \quad \beta_2 = 0, \tag{11.1} $$

$$ \eta_1 = -\eta_3 = \left(\frac{1}{2}A_1 h_2 + B_1\right)\left(\frac{G_2}{G_1} - 1\right), \quad \eta_2 = 0. \tag{11.2} $$

The factor κ^2 appearing in S_i is a shear coefficient introduced in the manner of Mindlin (1951). The proper choice of its value is discussed in Yu (1995).

Immovable in-plane boundary conditions at the edges Γ are considered throughout the paper, i.e.

$$ \Gamma : {}_2u_j^{(0)} = 0, \quad j = x, y, \tag{12} $$

and hence, moderately large lateral deflections may be considered by means of an approximation originally due to Berger (1955) in order to derive a simplified set of nonlinear governing equations. This approximation is based on the assumption that the elastic energy due to the second invariant in the extensional strain energy may disregarded as compared to the square of the first invariant without substantially affecting the response. In Berger's approach the in-plane forces are characterized by a time-variant isotropic force n, which is constant throughout the plate domain and is related to the deflection by the averaging integral, Wah (1963),

$$ n = \frac{D}{2\Omega}\int_\Omega (w_{,x}^2 + w_{,y}^2)\,\mathrm{d}\Omega, \quad D = \frac{2}{1-\nu}\sum_{i=1}^{3} A_i, \tag{13} $$

with Ω being the area of the plate domain. Following the arguments of Wu and Vinson (1969), n is not explicitly affected by the influence of shear.

The equations of motion are derived by considering the free-body diagram of an infinitesimal plate element, loaded by a given distributed transverse forcing function $p(x, y; t)$. Thereby, in a common approximation, both, the longitudinal as well as the rotatory inertia are neglected

$$_{\mathrm{i}}\ddot{u}_j^{(0)} = 0, \quad _{\mathrm{i}}\ddot{\psi}_j = 0, \tag{14}$$

thus, limiting the analysis to the lower frequency band of structural dynamics. Conservation of angular momentum about the x- and y-axes and conservation of momentum in x-, y- and z-direction render after some algebra the following equation of motion of the nonlinear plate problem in terms of the lateral deflection w,

$$K\Delta\Delta w + \frac{K}{S_\mathrm{e}} n\Delta\Delta w - n\Delta w - \mu\frac{K}{S_\mathrm{e}}\Delta\ddot{w} + \mu\ddot{w} = p - \frac{K}{S_\mathrm{e}}\Delta p. \tag{15}$$

Expression (15) can be interpreted as the equation of a homogeneous isotropic shear-deformable plate with bending stiffness K, effective shear stiffness S_e and mass per unit area μ, forced by the given lateral load p. The effective plate properties are given by

$$K = 2C_1 + C_2, \quad S_\mathrm{e} = \frac{\delta}{\alpha}K, \quad \mu = 2\rho_1 h_1 + \rho_2 h_2, \tag{16.1}$$

$$\alpha = \frac{2}{1-v}\left[B_1 h_2\left(\frac{G_2}{G_1} - 1\right) + 2C_1\frac{G_2}{G_1} + C_2\right]. \tag{16.2}$$

In (16.1) ρ_1, ρ_2 are the mass densities of the faces and the core, respectively.

The boundary conditions of a composite shear-deformable plate with hard hinged supports can be modeled in the form (see e.g. Adam, 2001)

$$\Gamma : w = 0, \quad _{\mathrm{i}}\psi_s = 0, \quad m_n = 0, \; i = 1, 2, 3, \tag{17}$$

where n, s are local Cartesian coordinates at boundary Γ with normal n pointing outwards. Furthermore, conservation of momentum in z-direction at Γ for a differential plate element renders

$$\Gamma : q_{n,n} + q_{s,s} + n\Delta w + p = 0. \tag{18}$$

Considering only polygonal contours $\Gamma : w = 0$ can be expressed by $w_{,s} = w_{,ss} = 0$ and $_{\mathrm{i}}\psi_s = 0$ may be replaced by $_{\mathrm{i}}\psi_{s,s} = 0$. Evaluating Eqs. (17) and (18) finally leads to two boundary conditions in w,

$$\Gamma : w = 0, \quad \Delta w + \frac{1}{S_\mathrm{e}} n\Delta w = -\frac{1}{S_\mathrm{e}}p. \tag{19}$$

3. An analogy to a homogeneous shear-deformable plate

According to Heuer (1992) for a composite beam and Adam (2001) for a linear composite plate a complete analogy to the nonlinear homogeneous shear-deformable plate on constrained in-plane supports is introduced by defining effective cross-sectional rotations $_{\mathrm{e}}\psi_x$, $_{\mathrm{e}}\psi_y$. The relation between $_{\mathrm{e}}\psi_x$, $_{\mathrm{e}}\psi_y$ and the actual plate deformation is found by equating the overall shear forces of the composite plate, Eq. (9), with the shear force of a corresponding homogenized shear-deformable plate

$$q_j = S_\mathrm{e}(_{\mathrm{e}}\psi_j + w_{,j}), \quad j = x, y, \tag{20}$$

and subsequent solution as

$$_{\mathrm{e}}\psi_j = \frac{\delta}{S_\mathrm{e}}\,_2\psi_j + \left(\frac{\delta}{S_\mathrm{e}} - 1\right)w_{,j}, \quad j = x, y. \tag{21}$$

Eq. (21) are substituted into (8) and also the overall bending and twisting moments are obtained in analogy to a homogeneous plate

$$m_j = K({}_{\mathrm{e}}\psi_{j,j} + v\,{}_{\mathrm{e}}\psi_{k,k}), \quad j = x,\ k = y \text{ and } k = x,\ j = y, \tag{22.1}$$

$$m_{xy} = \frac{1-v}{2}K({}_{\mathrm{e}}\psi_{x,y} + {}_{\mathrm{e}}\psi_{y,x}). \tag{22.2}$$

Consequently, by means of Eqs. (20)–(22) the equation of motion (15) can be separated to form a set of three equations

$$\mu\ddot{w} - S_{\mathrm{e}}(\Delta w + {}_{\mathrm{e}}\psi_{x,x} + {}_{\mathrm{e}}\psi_{y,y}) - n\Delta w = p, \tag{23.1}$$

$$K\left({}_{\mathrm{e}}\psi_{j,jj} + \frac{1-v}{2}{}_{\mathrm{e}}\psi_{j,kk} + \frac{1+v}{2}{}_{\mathrm{e}}\psi_{k,jk}\right) - S_{\mathrm{e}}({}_{\mathrm{e}}\psi_j + w_{,j}) = 0, \quad j = x,\ k = y \text{ and } k = x,\ j = y. \tag{23.2}$$

Eq. (23) describe the higher order problem of a geometric nonlinear composite plate with piecewise continuous in-plane displacement fields in full analogy to the lower order engineering theory of an isotropic homogeneous shear-deformable plate with horizontally constraint supports.

Boundary conditions for simply supported hard hinged straight edges are specified in analogy to a homogeneous elastic shear-deformable plate and they can be written as

$$\Gamma : w = 0, \quad {}_{\mathrm{e}}\psi_s = 0, \quad {}_{\mathrm{e}}\psi_{n,n} = 0. \tag{24}$$

The coupled set of equation (23) is solved together with the actual boundary conditions (24) for w, ${}_{\mathrm{e}}\psi_x$ and ${}_{\mathrm{e}}\psi_y$. Subsequently, the cross-sectional rotations of the core and the faces are to be determined. Decomposition of (21) yields the cross-sectional rotations of the core as

$${}_2\psi_j = \frac{S_{\mathrm{e}}}{\delta}{}_{\mathrm{e}}\psi_j - \left(1 - \frac{S_{\mathrm{e}}}{\delta}\right)w_{,j}, \quad j = x, y. \tag{25}$$

The cross-sectional rotations of the faces are calculated from (6)

$${}_i\psi_j = \frac{G_2}{G_i}({}_2\psi_j + w_{,j}) - w_{,j}, \quad i = 1, 3,\ j = x, y. \tag{26}$$

4. Evaluation of the nonlinear response

Eq. (23) are put formally into linear form by defining geometric nonlinearities in the operator as an additional equivalent lateral load acting on the homogeneous linear background plate, see e.g. Adam (2002). This additional equivalent lateral load is given by

$$q = n\Delta w. \tag{27}$$

Hence, efficient solution methods of the linear theory of flexural vibrations such as modal analysis become applicable. Since the distribution of q is not known in advance, but depends on the current state of deformation, the dynamic response has to be found incrementally by stepping the time and updating the equivalent lateral load. Thereby, the history of the load variables is divided into increments. The response increments δw, $\delta_{\mathrm{e}}\psi_x$, $\delta_{\mathrm{e}}\psi_y$ are formulated as a modal expansion

$$\delta w = \sum_{m=1,n=1}^{\infty} \delta Y_{mn}\Phi^{(mn)}, \tag{28.1}$$

$$\delta_e\psi_k = \sum_{m=1,n=1}^{\infty} \delta Y_{mn\,k}\Psi^{(mn)}, \quad k = x, y, \tag{28.2}$$

where $\Phi^{(mn)}$, ${}_x\Psi^{(mn)}$, ${}_y\Psi^{(mn)}$ denote ortho-normalized mode shapes. Substitution of these transformations into Eq. (23) and following the procedure of modal analysis (see e.g. Magrab, 1979) leads to a formally decoupled system of SDOF oscillator equations for the increments of the modal coordinates δY_{mn},

$$\delta\ddot{Y}_{mn} + 2\zeta_{mn}\omega_{mn}\delta\dot{Y}_{mn} + \omega_{mn}^2\delta Y_{mn} = \frac{1}{\mu}(\delta P_{mn} + \delta Q_{mn}). \tag{29}$$

In Eq. (29) structural damping is introduced via modal damping coefficients ζ_{mn}. Incorporation of viscous damping in the response is another convenient feature of the modal approach to the nonlinear problem. Furthermore, each mode can be damped individually, which seems to be more general than assuming a priori proportional Rayleigh damping. In Eq. (29) ω_{mn} is the mnth natural frequency and

$$\delta P_{mn} = \int_\Omega \Phi_{mn}\delta p \, d\Omega, \quad \delta Q_{mn} = \int_\Omega \Phi_{mn}\delta q \, d\Omega, \tag{30}$$

denote the generalized given and equivalent lateral load, respectively. The increments of the equivalent lateral load δq accounting for the nonlinear geometric effects become, Eqs. (13) and (27),

$$\delta q = n_a\delta\Delta w + \delta n[(\Delta w)_a + \delta\Delta w], \tag{31}$$

$$\delta n = \frac{D}{\Omega}\int_\Omega \left[(w_{,x})_a\delta w_{,x} + (w_{,y})_a\delta w_{,y} + \frac{1}{2}(\delta w_{,x}^2 + \delta w_{,y}^2)\right] d\Omega. \tag{32}$$

In (31) and (32) a subscript $()_a$ refers to variables at the beginning of the time step. The solution of the oscillator Eq. (29) is given by Duhamel's convolution integral. In order to secure convergent behavior within each time step a simple secant relation is applied, see Adam and Ziegler (1997) for details.

5. Application

In the following the dynamic response of a rectangular plate to time-harmonic excitation is examined. The plate of length a, width b and thickness h consists of three layers with layer thickness ratios $h_2/h = h_{1(3)}/h = 1/3$. The overall dimension of the considered plate is characterized by the factor $a/b = 2$. The mechanical properties of the faces and the core are specified through the ratio $G_{1(3)}/G_2 = 20$. Poisson's ratio is selected to be uniform to all layers, $\nu = 0.3$. A shear coefficient of $\kappa^2 = 1$ is considered. In particular, linear and nonlinear frequency response functions due to a uniformly distributed lateral load $p(x, y) = p_0$ are derived by sweeping the excitation frequency ν. At time $t = 0$ the time-harmonic load is subjected to the plate and a time history analysis is performed as described in Section 4. After decay of the transient response the maximum of the steady-state response is recorded for excitation frequencies in the neighborhood of the first and second natural frequency of the corresponding linear structure.

For the solution of the actually boundary value problem the natural frequencies ω_{mn} and mode shapes $\Phi^{(nm)}$, ${}_x\Psi^{(nm)}$ and ${}_y\Psi^{(nm)}$ of a rectangular shear-deformable plate are given by, see e.g. Magrab (1979),

$$\omega_{mn} = \alpha_{mn}\sqrt{\frac{KS_e}{\mu(S_e + K\alpha_{mn})}}, \quad \alpha_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2, \tag{33}$$

$$\Phi^{(mn)}(x, y) = A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{34.1}$$

$$ {}_x\Psi^{(mn)}(x,y) = B_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad {}_y\Psi^{(mn)}(x,y) = C_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \tag{34.2} $$

$$ A_{mn} = \frac{2}{\sqrt{ab}}, \quad B_{mn} = -\frac{A_{mn}a_m S_e}{K\alpha_{mn} + S_e}, \quad C_{mn} = -\frac{A_{mn}b_n S_e}{K\alpha_{mn} + S_e}, \ a_m = \frac{m\pi}{a}, \ b_n = \frac{n\pi}{b}. \tag{35} $$

The modal damping coefficients of all modes included in the computation are set to $\zeta_{mn} = 0.05$.

Fig. 2 shows non-dimensional amplitude functions of the lateral deflection at the center of the layered plate with a ratio thickness to length $h/a = 0.1$ for different non-dimensional load amplitudes $p^* = 2, 5, 10$. Thereby, p^* is defined as

$$ p^* = \frac{p_0 a^2 b}{K}. \tag{36} $$

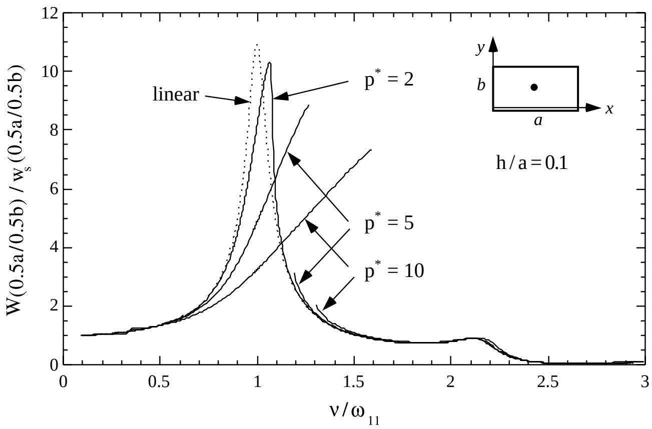

Fig. 2. Non-dimensional linear and nonlinear amplitude functions of the lateral deflection at the center of the plate for different loads p^*. Ratio of characteristic length over height $h/a = 0.1$. Stable branches of the response.

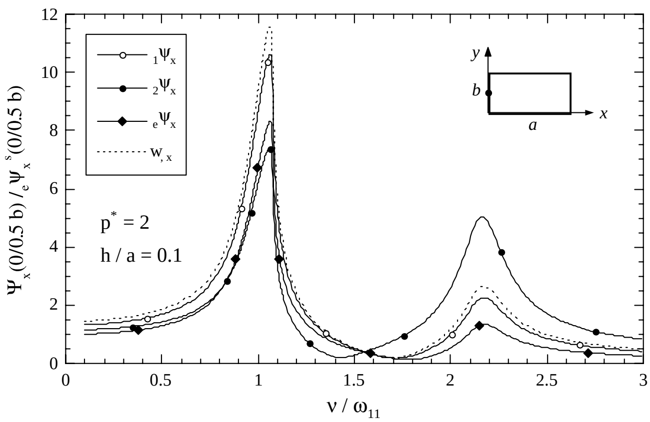

Fig. 3. Non-dimensional nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in x-direction and of the derivation of the deflection with respect to x at $(x/y = 0/0.5b)$ for a load of $p^* = 2$. Ratio of characteristic length over height $h/a = 0.1$.

The amplitude functions W are normalized by means of the corresponding static central deflection w_s, and the excitation frequency is related to the fundamental natural plate frequency ω_{11}. As expected the plate behaves in the neighborhood of the first natural frequency as a hard spring, and with increasing load amplitude the deviation to the linear response becomes more pronounced. The bending deformation of the resonance curves leads for $p^* = 5$ and 10 to multivalued amplitudes and the entire solution splits into stable and unstable branches. At those points where the tangent is vertical the well-known jump phenomenon occurs. However, in Fig. 2 only the stable portions of the response are displayed (since the outcomes derived by a time history analysis). The dynamic response close to the second symmetric natural plate frequency is almost not affected by the nonlinear terms in the plate equations. In addition, the correspondent linearized response is also presented in the same figure by a dashed line.

In Fig. 3 amplitude response functions of the individual cross-sectional rotations ${}_1\psi_x$ and ${}_2\psi_x$, the effective cross-sectional rotation ${}_e\psi_x$ and the derivative of the deflection with respect to ${}_xw_{,x}$ at point

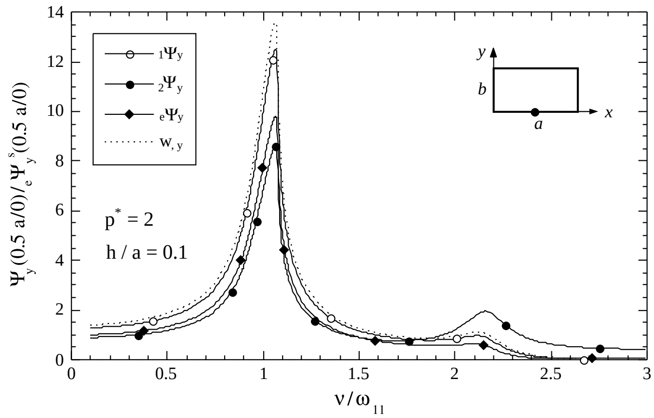

Fig. 4. Non-dimensional nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in y-direction and of the derivation of the deflection with respect to y at $(x/y = 0.5a/0)$ for a load of $p^* = 2$. Ratio of characteristic length over height $h/a = 0.1$.

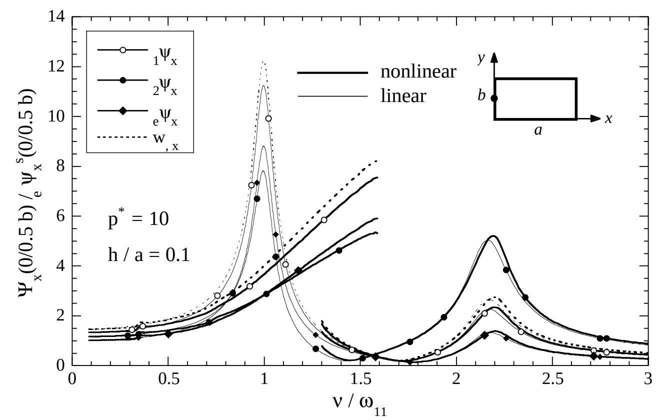

Fig. 5. Non-dimensional linear and nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in x-direction and of the derivation of the deflection with respect to x at $(x/y = 0/0.5b)$ for a load of $p^* = 10$. Ratio of characteristic length over height $h/a = 0.1$. Stable branches of the response.

$(x/y = 0/0.5b)$ are shown for a plate with the same thickness to length ratio $h/a = 0.1$. The x coordinate points in the direction of length a and its origin is in the lower corner on the left of the plate. All individual rotations are normalized by means of the corresponding static effective cross-sectional rotation ${}_e\psi_x^s$. A non-dimensional load amplitude of $p^* = 2$ is considered. It can be seen that the cross-sectional rotations of the core and the faces do not coincide. Particularly in the higher frequency range the deviation of the layer cross-sectional rotations is more pronounced. The amplitude response of the cross-sectional rotations of the faces ${}_1\psi_x$ are slightly overestimated by $w_{,x}$. Fig. 4 presents amplitude response functions of the cross-sectional rotations ${}_1\psi_y$, ${}_2\psi_y$, ${}_e\psi_y$ as well as of $w_{,y}$ at point $(x/y = 0.5a/0)$, and they are related to the static effective cross-sectional rotation ${}_e\psi_y^s$. The response in the neighborhood of the second symmetric mode is less pronounced compared to the results of Fig. 3. In Figs. 5 and 6 the stable branches for the same quantities as in Figs. 3 and 4 are shown for $p^* = 10$. For this non-dimensional load amplitude at $\omega_{11}/3$ the

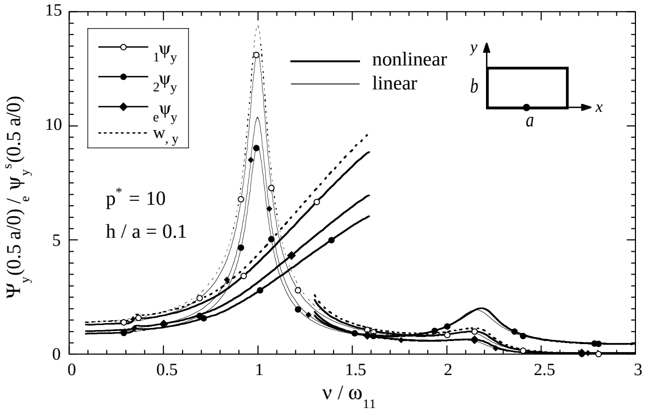

Fig. 6. Non-dimensional linear and nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in y-direction and of the derivation of the deflection with respect to y at $(x/y = 0.5a/0)$ for a load of $p^* = 10$. Ratio of characteristic length over height $h/a = 0.1$. Stable branches of the response.

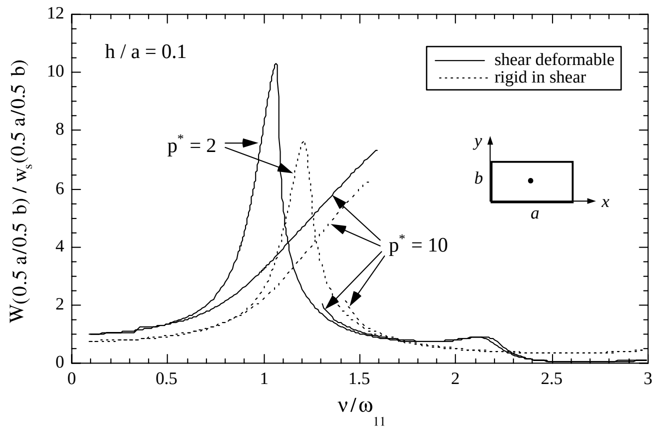

Fig. 7. Non-dimensional linear amplitude functions of the lateral deflection at the center of the plate for a loads $p^* = 2, 10$. Proposed (shear deformable) and classical (rigid in shear) plate theory. Ratio of characteristic length over height $h/a = 0.1$. Stable branches of the response.

influence of subharmonic resonance can be recognized by little spikes in the response. In addition, also the corresponding linearized quantities are set in contrast by thin lines in the same figures. Figs. 3–6 demonstrate that the nonlinear as well as linearized dynamic behavior of the considered plate ($h/a = 0.1$) cannot be predicted adequately by means of a first-order equivalent single-layer approximation, where a single cross-sectional rotations determines the horizontal displacement field for all layers.

In the following the influence of shear and plate thickness to length ratio on the harmonic steady-state response is studied. Figs. 7 and 8 compare non-dimensional amplitude functions of the central lateral deflection for plates with $h/a = 0.1$ and 0.05 and non-dimensional load amplitudes p^* of 2 and 10 considering and disregarding shear deformation. Thereby, for both considerations (shear deformable and rigid in shear) the amplitudes are related to the static deflection of the shear deformable plate and the excitation frequencies are normalized by means of the first natural frequency of the shear deformable composite plate.

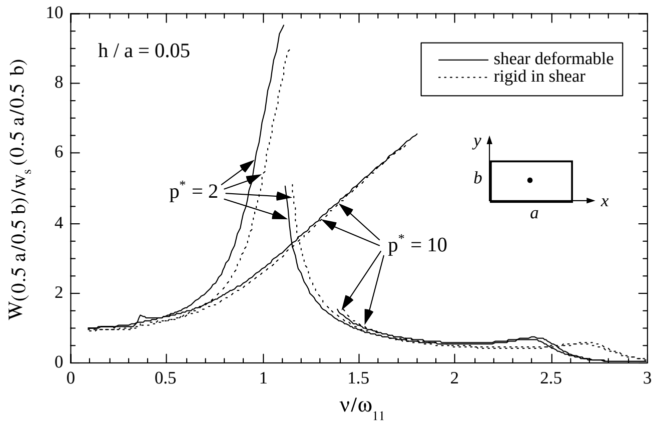

Fig. 8. Non-dimensional linear amplitude functions of the lateral deflection at the center of the plate for a load $p^* = 2, 10$. Proposed (shear deformable) and classical (rigid in shear) plate theory. Ratio of characteristic length over height $h/a = 0.05$. Stable branches of the response.

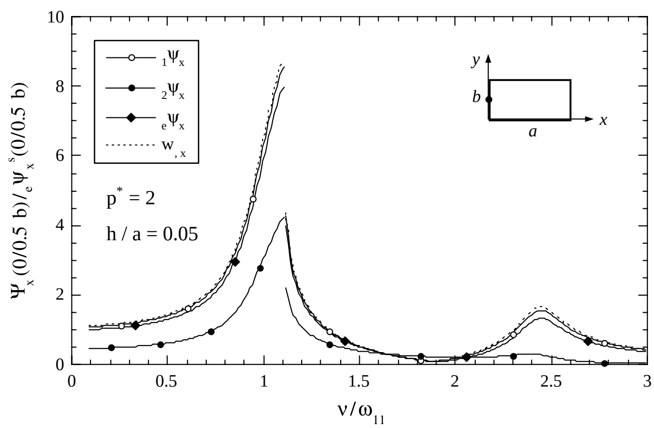

Fig. 9. Non-dimensional nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in x-direction and of the derivation of the deflection with respect to x at $(x/y = 0/0.5b)$ for a load of $p^* = 2$. Ratio of characteristic length over height $h/a = 0.05$. Stable branches of the response.

In Fig. 7, where results for h/a of 0.1 are presented, the shift of the peak response for the plate considered rigid in shear to the right hand side can be clearly seen. For a ratio of h/a of 0.05 the peak deflection in the neighborhood of the first natural frequency is much less affected by shear, in particular when the plate response is highly nonlinear ($p^* = 10$), see Figs. 8.

In the subsequent Figs. 9–12 the steady-state peak amplitude response of ${}_1\psi_x$, ${}_2\psi_x$, ${}_e\psi_x$, $w_{,x}$ at $(x/y = 0/0.5b)$ and ${}_1\psi_y$, ${}_2\psi_y$, ${}_e\psi_y$, $w_{,y}$ at $(x/y = 0.5a/0)$ for thickness to length ratios of $h/a = 0.05$ and 0.01 and load $p^* = 2$ is presented. Normalization as described for Figs. 3–6 is applied. Unlike to the results for $h/a = 0.1$ (Figs. 3–6) ${}_1\psi_x$, ${}_e\psi_x$, $w_{,x}$ on the one side ${}_1\psi_y$, ${}_e\psi_y$, $w_{,y}$ on the other side are almost in coincidence for $h/a = 0.05$, however, the cross-sectional rotations for the core ${}_2\psi_x$ and ${}_2\psi_y$ are different. These results indicate that a shear deformable theory is essential in order to predict horizontal displacements, even when

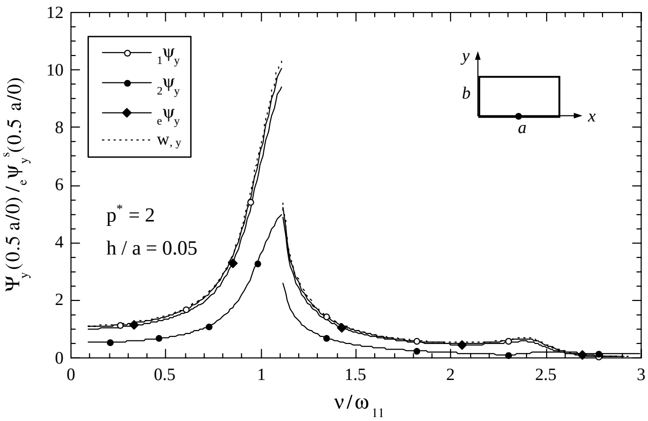

Fig. 10. Non-dimensional nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in y-direction and of the derivation of the deflection with respect to y at $(x/y = 0.5a/0)$ for a load of $p^* = 2$. Ratio of characteristic length over height $h/a = 0.05$. Stable branches of the response.

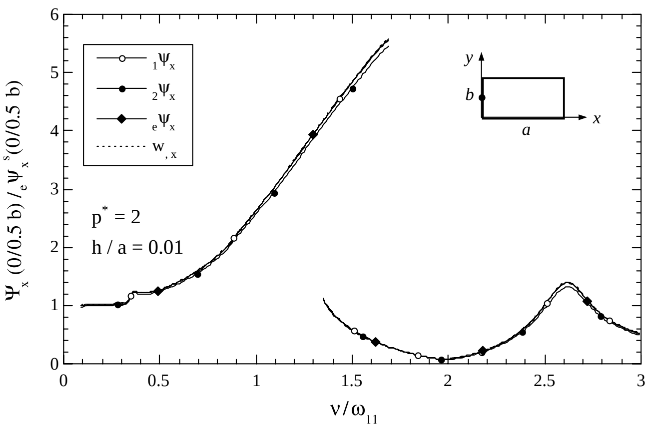

Fig. 11. Non-dimensional nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in x-direction and of the derivation of the deflection with respect to x at $(x/y = 0/0.5b)$ for a load of $p^* = 2$. Ratio of characteristic length over height $h/a = 0.01$. Stable branches of the response.

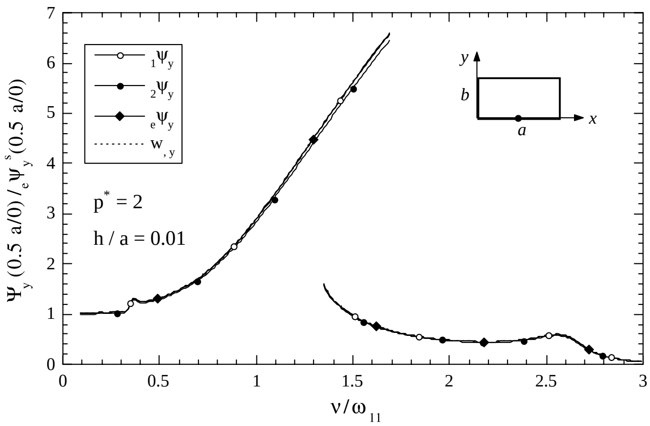

Fig. 12. Non-dimensional nonlinear amplitude functions of the layerwise and effective cross-sectional rotations in y-direction and of the derivation of the deflection with respect to y at $(x/y = 0.5a/0)$ for a load of $p^* = 2$. Ratio of characteristic length over height $h/a = 0.01$. Stable branches of the response.

the lateral deflection is not much effected by shear (see Fig. 8), since in the classical plate theory different cross-sectional rotations are not admitted. A comparison for a ratio of $h/a = 0.01$ reveals the diminishing influence of shear deformation on the dynamic response for thin plates, see Figs. 11 and 12. In the whole frequency range all quantities are almost in coincidence.

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References

Adam, C., 2001. Dynamic analysis of isotropic composite plates using a layerwise theory. Composite Structures 51, 427–437.

Adam, C., 2002. Moderately large vibrations of imperfect elastic–plastic composite beams with thick layers. International Journal of Acoustics and Vibration 7, 11–20.

Adam, C., Ziegler, F., 1997. Moderately large forced oblique vibrations of elastic–viscoplastic deteriorating slightly curved beams. Archive of Applied Mechanics 67, 375–392.

Berger, H.M., 1955. A new approach to the analysis of large deflection of plates. Journal of Applied Mechanics 22, 465–472.

Di Taranto, R.A., 1965. Theory of vibratory bending for elastic and viscoelastic layered finite-length beams. Journal of Applied Mechanics 32, 881–886.

Durocher, L.L., Solecki, R., 1976. Harmonic vibrations of isotropic, elastic, and elastic/visco-elastic three-layered plates. Journal of Acoustic Society of America 60, 105–112.

Gordaninejad, F., Bert, C.W., 1989. A new theory for bending of thick sandwich beams. International Journal of Mechanical Sciences 31, 925–934.

Heuer, R., 1992. Static and dynamic analysis of transversely isotropic, moderately thick sandwich beams by analogy. Acta Mechanica 91, 1–9.

Heuer, R., 1994. Large flexural vibrations of thermally stressed layered shallow shells. Nonlinear Dynamics 5, 25–38.

Heuer, R., Ziegler, F., 1998. Nonlinear resonance phenomena of panel-type structures. Computer and Structures 67, 65–70.

Heuer, R., Irschik, H., Fotiu, P., Ziegler, F., 1992. Nonlinear flexural vibrations of layered plates. International Journal of Solids and Structures 29, 1813–1818.

Heuer, R., Irschik, H., Ziegler, F., 1997. Thermo-piezoelectric vibrations of three-layer elastic plates. In: Hetnarski, R.B., Noda, N. (Eds.), Proceedings of the Second International Symposium on Thermal Stresses and Related Topics, Rochester, NY, 8–11 June 1997. RIT, pp. 451–454.

Leissa, A.W., 1981. Advances in vibration, buckling and postbuckling studies on composite plates. In: Marshall, I.H. (Ed.), Composite Structures. Applied Science Publishers, London, pp. 312–334.

Magrab, E.B., 1979. Vibrations of Elastic Structural Members. Sijthoff and Noordhoff, Alphen aan den Rijn.

Mindlin, R.D., 1951. Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates. Journal of Applied Mechanics 18, 31–38.

Reddy, J.N., 1984. A simple higher-order theory for laminated composite plates. Journal of Applied Mechanics 51, 745–752.

Reddy, J.N., 1997. Mechanics of Laminated Composite Plates. Theory and Analysis. CRC Press, Boca Raton.

Reddy, J.N., Chao, W.C., 1981. Large-deflection and large-amplitude free vibrations of laminated composite-material plates. Computer and Structures 13, 341–347.

Ribeiro, P., Petyt, M., 1999. Non-Linear vibration of composite laminated plates by the hierarchical finite element method. Composite Structures 46, 197–208.

Sathyamoorthy, M., 1998. Nonlinear Analysis of Structures. CRC Press, Boca Raton.

Shi, Y., Lee, R.Y.Y., Mei, C., 1997. Finite element method for nonlinear free vibrations of composite plates. AIAA Journal 35, 159–166.

Suzuki, K., Shikanai, G., Leissa, A.W., 1996. Free vibrations of laminated composite non-circular thick cylindrical shells. International Journal of Solids and Structures 33, 4079–4100.

Swift, G.W., Heller, R.A., 1974. Layered beam analysis. Journal of the Engineering Mechanics Division ASCE 100, 267–282.

von Karman, T., 1910. Festigkeitsprobleme im Maschinenbau. In: Encyklopaedie der mathematischen Wissenschaften IV. Teubner, Leipzig.

Wah, T., 1963. Large amplitude flexural vibration of rectangular plates. International Journal of Mechanical Science 5, 425–438.

Whitney, J.M., Pagano, N.J., 1970. Shear deformation in heterogeneous anisotropic plates. Journal of Applied Mechanics 37, 1031–1036.

Wu, C.-I., Vinson, J.R., 1969. Influences of large amplitudes, transverse shear deformation, and rotatory inertia on lateral vibrations of transversely isotropic plates. Journal of Applied Mechanics 36, 254–260.

Yan, M.-J., Dowell, E.H., 1972. Governing equations for vibrating constrained-layer damping sandwich plates and beams. Journal of Applied Mechanics 39, 1041–1046.

Yu, Y.-Y., 1995. Vibrations of Elastic Plates. Springer, New York.

Ziegler, F., 1998. Mechanics of Solids and Fluids, Second ed Springer, New York (2nd reprint).